LECTURE NOTE

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1. Preliminaries draft

Definition 1.1. A *computable metric space* is a triple $(X,\,\rho,\,\mathcal{S})$, where

(i) (X,ρ) is a separable metric space;
(ii) $\mathcal{S}=\{s_n : n\in\mathbb{N}\}$ is a countable dense subset of X; and
(iii) there exists an algorithm which, on input $i,\,j,\,m\in\mathbb{N}$, outputs $y_{i,j,m}\in\mathbb{Q}$ satisfying $\bigl|y_{i,j,m}-\rho\bigl(s_i,s_j\bigr)\bigr|<2^{-m}$.

The points in $\mathcal{S}$ are said to be *ideal*. Due to the existence of computable bijection between $\mathbb{N}^3$ and $\mathbb{N}$, there exists an effective enumeration $\{B_l\}_{l\in\mathbb{N}}$ of the set $\{B(s_i,j/k) : i,\,j,\,k\in\mathbb{N}\}$ of balls with rational radii centered at points in $\mathcal{S}$. Specifically, there exists an algorithm that, given an input $l\in\mathbb{N}$, outputs the lower index of the ideal center and the rational radius of the ball B_l. These balls are called the *ideal balls* in $(X,\,\rho,\,\mathcal{S})$. We fix such an effective enumeration of ideal balls and call it the effective enumeration of ideal balls in $(X,\,\rho,\,\mathcal{S})$.

Definition 1.2. In a computable metric space $(X,\,\rho,\,\mathcal{S})$, an open set $U\subseteq X$ is called *lower semi-computable* if there is a computable function $f\colon\mathbb{N}\to\mathbb{N}$ such that $U=\bigcup_{n\in\mathbb{N}}B_{f(n)}$. Moreover, a sequence $\{U_i\}_{i\in\mathbb{N}}$ of lower semi-computable open sets is called *a sequence of uniformly lower semi-computable open sets* if there is a computable function $f\colon\mathbb{N}^2\to\mathbb{N}$ such that $U_i=\bigcup_{n\in\mathbb{N}}B_{f(i,n)}$ for each $i\in\mathbb{N}$.

Definition 1.3. Let $(X,\,\rho,\,\mathcal{S})$ and $(X',\,\rho',\,\mathcal{S}')$ be computable metric spaces with $\mathcal{S}=\{s_i\}_{i\in\mathbb{N}}$ and $\mathcal{S}'=\bigl\{s'_i\bigr\}_{i\in\mathbb{N}}$, and let C be a subset of X. A function $f\colon X\to X'$ is said to be *computable on C* if there exists an algorithm that for each $x\in X$ and each $n\in\mathbb{N}$, on input $n\in\mathbb{N}$ and an oracle φ for $x\in C$, outputs $m\in\mathbb{N}$ satisfying $\rho'\bigl(s'_m,f(x)\bigr)<2^{-n}$. Moreover, a sequence $\{f_i\}_{i\in\mathbb{N}}$ of functions $f_i\colon X\to X'$ is called a *sequence of uniformly computable functions on C* if there exists an algorithm that for each $x\in X$, each $i\in\mathbb{N}$, and each $n\in\mathbb{N}$, on input $i,\,n\in\mathbb{N}$, and an oracle φ for $x\in C$, outputs $m\in\mathbb{N}$ satisfying $\rho'\bigl(s'_m,f_i(x)\bigr)<2^{-n}$. As a convention, we say that f is computable if f is computable on X.

Proposition 1.4. *Let $(X,\,\rho,\,\mathcal{S})$ and $(X',\,\rho',\,\mathcal{S}')$ be two computable metric spaces, C be a subset of X, and $\bigl\{B'_i\bigr\}_{i\in\mathbb{N}}$ be the effective enumeration of ideal balls in $(X',\,\rho',\,\mathcal{S}')$. Assume that $\{f_i\}_{i\in\mathbb{N}}$ is a sequence of functions $f_i\colon X\to X'$. Then $\{f_i\}_{i\in\mathbb{N}}$ is a sequence of uniformly computable functions on C if and only if there exists a sequence $\bigl\{U_{i,j}\bigr\}_{i,j\in\mathbb{N}}$ of uniformly lower semi-computable open*

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sets in the computable metric space $(X,\,\rho,\,\mathcal{S})$ *satisfying that* $f_j^{-1}\big(B_i'\big)\cap C=U_{i,j}\cap C$ *for each pair of* $i,\,j\in\mathbb{N}$.

Proof. Write $\mathcal{S}=\{s_i\}_{i\in\mathbb{N}}$ and $\mathcal{S}'=\big\{s_i'\big\}_{i\in\mathbb{N}}$. Now we assume that $\{f_i\}_{i\in\mathbb{N}}$ is a sequence of uniformly computable functions on C and show that there exists a sequence $\big\{U_{i,j}\big\}_{i,j\in\mathbb{N}}$ of uniformly lower semi-computable open sets in the computable metric space $(X,\,\rho,\,\mathcal{S})$ satisfying $f_j^{-1}(B_i')\cap C=U_{i,j}\cap C$ for each pair of $i,\,j\in\mathbb{N}$. For each $q\in\mathbb{N}$, we say that a sequence $\{p_i\}_{i=1}^{q}$ of integers is admissible in the computable metric space $(X,\,\rho,\,\mathcal{S})$ if $\rho\big(s_{p_{i+1}},s_{p_i}\big)<2^{-i-1}$ for each $i\in\mathbb{N}\cap[1,q-1]$. By Definition 1.1 (iii), we can check whether a given sequence of finitely many integers is admissible. Hence, by enumerating all the sequences of finitely many integers, it is not difficult to obtain an effective enumeration $\{P_i\}_{i\in\mathbb{N}}$ of all possible admissible sequences in $(X,\,\rho,\,\mathcal{S})$. Moreover, for each admissible sequence $P=\{p_i\}_{i=1}^{q}$, we can define a corresponding function $\varphi_P\colon\mathbb{N}\to\mathbb{N}$ as follows:

$$\varphi_P(i)\coloneqq\begin{cases}p_i & \text{if } 1\leqslant i\leqslant q;\\ p_q & \text{if } i\geqslant q+1\end{cases}\qquad\text{for each } i\in\mathbb{N}$$

is an oracle for the point $s_{p_q}\in X$.

Since $\{f_i\}_{i\in\mathbb{N}}$ is a sequence of uniformly computable functions on C, there exists an algorithm $M(\cdot,\cdot,\cdot)$ that satisfies that for each $x\in C$, each $n\in\mathbb{N}$, each $i\in\mathbb{N}$, and each oracle φ for x, $M(i,n,\varphi)$ outputs $m\in\mathbb{N}$ satisfying that $\rho'(s_m',f_i(x))<2^{-n}$. We enumerate $\mathbb{N}\times\mathbb{N}$ by $\{(a_u,n_u)\}_{u\in\mathbb{N}}$ effectively. Now we design an algorithm $M'(\cdot,\cdot)$ which, for each pair of $i,\,j\in\mathbb{N}$, on input $i,\,j\in\mathbb{N}$, outputs a sequence $\{c_{i,j,k}\}_{k\in\mathbb{N}}$ of integers and a sequence $\{r_{i,j,k}\}_{k\in\mathbb{N}}$ of rational numbers satisfying that $f_j^{-1}(B_i')\cap C=\bigcup_{k\in\mathbb{N}}\big(B_\rho\big(s_{c_{i,j,k}},r_{i,j,k}\big)\cap C\big)$ for each $i,\,j\in\mathbb{N}$ as follows.

Begin

(i) Read in the integers i and j.

(ii) Set u and k both to be 1, and $\text{flag}_i=0$ for each $i\in\mathbb{N}$.

(iii) **While** $u\geqslant 1$ **do**

  (1) Run the algorithm $M\big(j,n_u,\varphi_{P_{a_u}}\big)$.

  (2) Set v to be 1.

  (3) **While** $1\leqslant v\leqslant u$ **do**

    (A) If

      (a) flag_v equals to 0,

      (b) the algorithm $M\big(j,n_v,\varphi_{P_{a_v}}\big)$ halts and outputs $m_v\in\mathbb{N}$ satisfying that

$$B_{\rho'}\big(s_{m_v}',2^{-n_v}\big)\subseteq B_i'$$

      (the algorithm $M\big(j,n_v,\varphi_{P_{a_v}}\big)$ terminates after finitely many steps, and hence the oracle $\varphi_{P_{a_v}}$ is only quired up to some finite precision 2^{-w_v}),

    then

      (a′) the algorithm $M'(i,j)$ outputs $c_{i,j,k}\coloneqq\varphi_{P_{a_v}}(w_v)$ and $r_{i,j,k}\coloneqq 2^{-w_v}$,

      (b′) set flag_v to be 1 and k to be $k+1$.

    (B) Set v to be $v+1$.

  (4) Set u to be $u+1$.

End

Now we fix an pair of $i,\,j\in\mathbb{N}$, and verify that $f_j^{-1}(B_i')\cap C=\bigcup_{k\in\mathbb{N}}\big(B_\rho\big(s_{c_{i,j,k}},r_{i,j,k}\big)\cap C\big)$.

First, we fix an integer k and show that $B_\rho\big(s_{c_{i,j,k}}, r_{i,j,k}\big) \cap C \subseteq f_j^{-1}(B'_i)$. By Step (iii) (3) (A) of the algorithm $M'(i,j)$, we obtain that $c_{i,j,k} = \varphi_{P_{a_v}}(w_v)$ and $r_{i,j,k} = 2^{-w_v}$ for some $v \in \mathbb{N}$ with $B_{\rho'}\big(s'_{m_v}, 2^{-n_v}\big) \subseteq B'_i$. Here m_v is the output of the algorithm $M\big(j, n_v, \varphi_{P_{a_v}}\big)$. Note that $\mathcal{S} = \{s_i\}_{i\in\mathbb{N}}$ is dense in X. It is not hard to see that, for each $x \in B_\rho\big(s_{c_{i,j,k}}, r_{i,j,k}\big) \cap C$, there is a valid oracle $\widetilde{\varphi}_x$ that agrees with $\varphi_{P_{a_v}}$ up to precision 2^{-w_v}. Thus for each $x \in B_\rho\big(s_{c_{i,j,k}}, r_{i,j,k}\big) \cap C$, $M(j,\, n_v,\, \widetilde{\varphi}_x)$ outputs the same answer m_v and hence, we must have $f_j(x) \in B_{\rho'}\big(s'_{m_v}, 2^{-n_v}\big) \subseteq B'_i$. Then we have $f_j\big(B_\rho\big(s_{c_{i,j,k}}, r_{i,j,k}\big) \cap C\big) \subseteq B'_i$. Therefore, we obtain that $\bigcup_{k\in\mathbb{N}}\big(B_\rho\big(s_{c_{i,j,k}}, r_{i,j,k}\big) \cap C\big) \subseteq f_j^{-1}(B'_i) \cap C$.

Next, we establish that $\bigcup_{k\in\mathbb{N}}\big(B_\rho\big(s_{c_{i,j,k}}, r_{i,j,k}\big) \cap C\big) \supseteq f_j^{-1}(B'_i) \cap C$. Now we fix an point $x \in f_j^{-1}(B'_i) \cap C$, and show that $x \in B_\rho\big(s_{c_{i,j,k}}, r_{i,j,k}\big)$ for some $k \in \mathbb{N}$. Indeed, since $f_j(x) \in B'_i$, there exists $n(x) \in \mathbb{N}$ satisfying that $B_{\rho'}\big(f_j(x), 2^{-n(x)+1}\big) \subseteq B'_i$. Note that $\mathcal{S}$ is dense in X. It is not hard to see that, for $x \in C$, there is a valid oracle $\overline{\varphi}_x$ that satisfies that $\{\overline{\varphi}_x(v)\}_{v=1}^{q}$ is an admissible sequence for each $q \in \mathbb{N}$. Note that $x \in C$. Then the algorithm $M(j, n(x), \overline{\varphi}_x)$ will halt eventually. Assume that the output of the algorithm $M(j, n(x), \overline{\varphi}_x)$ is $m(x)$. Then $\rho'\big(s'_{m(x)}, f_j(x)\big) < 2^{-n(x)}$. Hence, $B_{\rho'}\big(s'_{m(x)}, 2^{-n(x)}\big) \subseteq B_{\rho'}\big(f_j(x), 2^{-n(x)+1}\big) \subseteq B'_i$. Assume that the oracle $\overline{\varphi}_x$ is only quired up to the precision $2^{-w(x)}$ by the algorithm $M(j, n(x), \overline{\varphi}_x)$. Denote the sequence $\{\overline{\varphi}_x(v)\}_{v=1}^{w(x)}$ by $Q(x)$. Then $Q(x)$ is an admissible sequence and the oracle $\varphi_{Q(x)}$ agrees with $\overline{\varphi}_x$ up to precision $2^{-w(x)}$. Thus $M\big(j,\, n(x),\, \varphi_{Q(x)}\big)$ outputs the same answer $m(x) \in \mathbb{N}$ as $M(j,\, n(x),\, \overline{\varphi}_x)$. Since $Q(x)$ is an admissible sequence, we will run the algorithm $M\big(j,\, n(x),\, \varphi_{Q(x)}\big)$ in Step (iii) (1) of the algorithm $M'(i,j)$. Since $B_{\rho'}\big(s'_{m(x)}, 2^{-n(x)}\big) \subseteq B'_i$, in Step (iii) (3) (A) of the algorithm $M'(i,j)$, $M'(i,j)$ will output $c_{i,j,k} = \varphi_{Q(x)}(w(x)) = \overline{\varphi}_x(w(x))$ and $r_{i,j,k} = 2^{-w(x)}$ for some $k \in \mathbb{N}$. Note that $\overline{\varphi}_x$ is an oracle for x. Then we have $x \in B_\rho\big(s_{\overline{\varphi}_x(w(x))}, 2^{-w(x)}\big) = B_\rho\big(s_{c_{i,j,k}}, r_{i,j,k}\big)$.

Hence, $f_j^{-1}(B'_i) \cap C = \big(\bigcup_{k\in\mathbb{N}} B_\rho\big(s_{c_{i,j,k}}, r_{i,j,k}\big)\big) \cap C$ for each pair of $i,\, j \in \mathbb{N}$. Note that by the existence of the algorithm $M'(\cdot,\cdot)$, we have $\big\{B_\rho\big(s_{c_{i,j,k}}, r_{i,j,k}\big) : i,\, j,\, k \in \mathbb{N}\big\}$ is a sequence of uniformly lower semi-computable open sets in the computable metric space $(X,\, \rho,\, \mathcal{S})$. From Definition 1.2, by constructing a computable bijection between $\mathbb{N}^3$ and $\mathbb{N}^2$, it is not hard to derive that $\big\{\bigcup_{k\in\mathbb{N}} B_\rho\big(s_{c_{i,j,k}}, r_{i,j,k}\big) : i,\, j \in \mathbb{N}\big\}$ is a sequence of uniformly lower semi-computable open sets in the computable metric space $(X,\, \rho,\, \mathcal{S})$.

Finally, we assume that there exists a sequence $\{U_{i,j} : i,\, j \in \mathbb{N}\}$ of uniformly lower semi-computable open sets in the computable metric space $(X,\, \rho,\, \mathcal{S})$ satisfying that $f_j^{-1}(B'_i) \cap C = U_{i,j} \cap C$ for each pair of $i,\, j \in \mathbb{N}$ and establish that $\{f_i\}_{i\in\mathbb{N}}$ is a sequence of uniformly computable functions on C. Now we fix an oracle φ_x of a point $x \in C$ and a pair of $i,\, n \in \mathbb{N}$. By Definition 1.3, it suffices to compute an integer m satisfying that $\rho'(s'_m, f_i(x)) < 2^{-n}$, i.e., $x \in f_i^{-1}(B_{\rho'}(s'_m, 2^{-n}))$.

Indeed, by hypotheses, we can compute a sequence $\{U_m\}_{m\in\mathbb{N}}$ of lower semi-computable open sets satisfying that $f_i^{-1}(B_{\rho'}(s'_m, 2^{-n})) \cap C = U_m \cap C$ for each $m \in \mathbb{N}$. Note that $x \in C$. Then $x \in f_i^{-1}(B_{\rho'}(s'_m, 2^{-n}))$ if and only if $x \in U_m$, i.e., $B_\rho\big(s_{\varphi_x(t)}, 2^{-t}\big) \subseteq U_m$ for some $t \in \mathbb{N}$. By the uniform lower semi-computable openness of the sequence $\{U_m\}_{m\in\mathbb{N}}$, it is not hard to construct an algorithm which, on input $m \in \mathbb{N}$, halts if and only if $x \in U_m$. Note that $\mathcal{S} = \{s_m\}_{m\in\mathbb{N}}$ is dense in X. Then there exists an integer m satisfying that $x \in U_m$. Therefore, we can find an integer $m \in \mathbb{N}$ such that $x \in U_m$ for each $x \in X$. Therefore, we establish that $\{f_i\}_{i\in\mathbb{N}}$ is a sequence of uniformly computable functions on C. □

Lemma 1.5. *There exists an algorithm that satisfies the following property:*

For each $m \in \mathbb{N}$, each $n \in \mathbb{N}$, and each complex polynomial p of degree m, this algorithm outputs a sequence $\{q_i\}_{i=1}^{m}$ of integers satisfying that if $x_1,\, x_2, \dots,\, x_m$ are all the zeros of the map p (counting with multiplicity), then there exists a permutation σ on $\{1,\, 2, \dots,\, m\}$ such that

$\sigma\big(u_{q_{\sigma(i)}}, x_i\big) < 2^{-n}$ for each integer $1 \leqslant i \leqslant m$, where $\{u_j\}_{j\in\mathbb{N}}$ is the effective enumeration of the set $\mathbb{Q}(\widehat{\mathbb{C}})$, after we input the following data in this algorithm:

(i) *an algorithm $\mathcal{A}_p$ computing all the coefficients of the polynomial p,*

(ii) *the integer n.*

Proof. Let $\{s_i\}_{i\in\mathbb{N}}$ be an effective enumeration of the set $\{a + b\mathbf{i} : a, b \in \mathbb{Q}\}$. Now we design an algorithm $M(\,\cdot\,,\cdot\,)$ satisfying the following property:

For each polynomial Q, there exists a zero z_0 of Q satisfying that for each $m \in \mathbb{N}$, $M\big(\mathcal{A}_Q, m\big)$ outputs a point $l_m \in \mathbb{Q}\big(\widehat{\mathbb{C}}\big)$ with $\sigma(l_m, z_0) < 2^{-m}$ after we input an algorithm $\mathcal{A}_Q$ computing all the coefficients of the polynomial Q and the integer m.

First, we use the algorithm $\mathcal{A}_Q$ to compute the sequence $\{Q'(s_i)\}$ and select a subsequence $\{\tilde{s}_i\}_{i\in\mathbb{N}}$ of $\{s_i\}_{i\in\mathbb{N}}$ of all the ideal points $\tilde{s}_i$ with $Q'(\tilde{s}_i) \neq 0$. Then we define two sequences $\{\gamma(Q,i)\}_{i\in\mathbb{N}}$ and $\{\beta(Q,i)\}_{i\in\mathbb{N}}$ by

$$\gamma(Q,i) := \sup_{k\geqslant 2}\left|\frac{Q^{(k)}(\tilde{s}_i)}{k!Q'(\tilde{s}_i)}\right|^{\frac{1}{k-1}} \quad \text{and} \quad \beta(Q,i) := \left|\frac{Q(\tilde{s}_i)}{Q'(\tilde{s}_i)}\right|. \tag{1.1}$$

Since there exist finitely many roots for the rational map Q' and $\{s_i\}_{i\in\mathbb{N}}$ is dense in $\mathbb{C}$, $\{\tilde{s}_i\}_{i\in\mathbb{N}}$ is also dense in $\mathbb{C}$. Combining with the fact that $\beta(Q,\xi) = 0$ for each root $\xi \in \mathbb{C}$ of Q, we can enumerate the sequence $\{\tilde{s}_i\}_{i\in\mathbb{N}}$ and find $i_0 \in \mathbb{N}$ with $\alpha(Q,i_0) := \beta(Q,i_0)\gamma(Q,i_0) < \alpha_0$ (here we can select $\alpha_0 := 0.03$, see Remark 6 of [BCSS98, Section 8.2]). Next, compute an integer k_m with

$$k_m > \log_2(m + 4 + \log_2(\beta(Q,i_0))). \tag{1.2}$$

Hence, by Theorem 2 of [BCSS98, Section 8.2], there exists a zero $z_0 \in \mathbb{C}$ of Q satisfying that

$$\big|N_Q^t(\tilde{s}_{i_0}) - z_0\big| \leqslant \frac{|\tilde{s}_{i_0} - z_0|}{2^{2^t-1}} \leqslant \frac{2\beta(Q,i_0)}{2^{2^t-1}} \quad \text{for each } t \in \mathbb{N}.$$

Here $N_Q(z) := z - \frac{Q(z)}{Q'(z)}$ for each $z \in \mathbb{C}$. Combining with (1.2), this implies that

$$\big|N_Q^{k_m}(\tilde{s}_{i_0}) - z_0\big| \leqslant \frac{2\beta(Q,i_0)}{2^{2^{k_m}-1}} < \frac{2\beta(Q,i_0)}{2^{m+3+\log_2(\beta(Q,i_0))}} = \frac{1}{2^{m+2}}. \tag{1.3}$$

Finally, we use the algorithm $\mathcal{A}_Q$ to compute and output a point $l_m \in \mathbb{Q}\big(\widehat{\mathbb{C}}\big)$ with $\big|l_m - N_Q^{k_m}(\tilde{s}_{i_0})\big| < 2^{-m-2}$. It follows immediately from the definition of the chordal metric σ on $\widehat{\mathbb{C}}$ (see Section **??**) that $\sigma(z,w) \leqslant 2|z-w|$ for each pair of z, $w \in \mathbb{C}$. Hence, by (1.3),

$$\sigma(l_m, z_0) \leqslant 2|l_m - z_0| \leqslant 2\big(\big|l_m - N_Q^{k_m}(\tilde{s}_{i_0})\big| + \big|N_Q^{k_m}(\tilde{s}_{i_0}) - z_0\big|\big) < 2^{-m}.$$

So far we have designed the algorithm $M(\cdot,\cdot)$.

Next, we come back to the proof of the original statement. Fix an integer n and a complex polynomial p of degree n. First, we can use the algorithm $M\big(\mathcal{A}_p, \cdot\big)$ to compute a zero of the polynomial p, say z_0. Then we consider the map $\overline{p}(z) := \frac{p(z)}{z-z_0}$. Since $p(z_0) = 0$, $\overline{p}$ is a polynomial of degree $n-1$. Now we claim that we can compute all the coefficients of the polynomial $\overline{p}$ from the point z_0 and all the coefficients of the polynomial p. Indeed, if $p(z) = \sum_{i=0}^{n} a_i z^i$ and $\overline{p}(z) = \sum_{i=0}^{n-1} b_i z^i$, then it is not hard to see that $b_i = a_{i+1} + z_0 b_{i+1}$ for each integer $0 \leqslant i \leqslant n-1$, where b_n is set to be 0. Hence, we obtain an algorithm $\mathcal{A}_{\overline{p}}$ computing all the coefficients of $\overline{p}$. Then we can use the algorithm $M\big(\mathcal{A}_{\overline{p}}, \cdot\big)$ to compute a zero of the polynomial $\overline{p}$, i.e., a new zero of the polynomial p. Therefore, we can compute all the zeros of p (counting with multiplicity) recursively. □

References

[BD23] Bianchi, F. and Dinh, T.-C., Equilibrium states of endomorphisms of $\mathbb{P}^k$ I: existence and properties, *J. Math. Pures Appl.* 172 (2023), 164–201.

[BD24] Bianchi, F. and Dinh, T.-C., Equilibrium States of Endomorphisms of $\mathbb{P}^k$: Spectral Stability and Limit Theorems, *Geom. Funct. Anal.* 34 (2024), 1006–1051.

[BBRY11] Binder, I., Braverman, M., Rojas, C., and Yampolsky, M., Computability of Brolin–Lyubich measure, *Comm. Math. Phys.* 308 (2011), 743–771.

[BGRY22] Binder, I., Glucksam, A., Rojas, C., and Yampolsky, M., Computability in Harmonic Analysis, *Found. Comput. Math.* 22 (2022), 849–873.

[BHLZ24] Binder, I., He, Q., Li, Z., and Zhang, Y., *On computability of equilibrium states*, 2024, arXiv: `2311.09374 [math.DS]`.

[BRY14] Binder, I., Rojas, C., and Yampolsky, M., Computable caratheodory theory, *Adv. Math.* 265 (2014), 280–312.

[BCSS98] Blum, L., Cucker, F., Shub, M., and Smale, S., *Complexity and Real Computation*, 1998.

[BM10] Bonk, M. and Meyer, D., *Expanding Thurston Maps*, 2010, arXiv: `1009.3647 [math.DS]`.

[BM17] Bonk, M. and Meyer, D., *Expanding Thurston Maps*, vol. 225, Amer. Math. Soc., Providence, RI, 2017.

[BBY12] Bonnot, S., Braverman, M., and Yampolsky, M., Thurston equivalence to a rational map is decidable, *Mosc. Math. J.* 12 (2012), 747–763, 884.

[Bow75] Bowen, R., Equilibrium states and the ergodic theory of Anosov diffeomorphisms, *Lecture Notes in Math.* 470 (1975), 1–17.

[BY06] Braverman, M. and Yampolsky, M., Non-computable Julia sets, *J. Amer. Math. Soc.* 19 (2006), 551–578.

[Bra23] Braverman, M., Communication and information complexity, *Proc. Internat. Congr. Math. (2022)*, Vol. I, EMS Press, Berlin, 2023, 284–320.

[BY09] Braverman, M. and Yampolsky, M., *Computability of Julia sets*, vol. 23, Springer, 2009.

[Bro65] Brolin, H., Invariant sets under iteration of rational functions, *Ark. Mat.* 6 (1965), 103–144.

[BDWY22] Burr, M., Das, S., Wolf, C., and Yang, Y., Computability of topological pressure on compact shift spaces beyond finite type, *Nonlinearity* 35 (2022), 4250–4282.

[BS03] Buzzi, J. and Sarig, O., Uniqueness of equilibrium measures for countable Markov shifts and multidimensional piecewise expanding maps, *Ergodic Theory Dynam. Systems* 23 (2003), 1383–1400.

[Can94] Cannon, J. W., The combinatorial Riemann mapping theorem, *Acta Math.* 173 (1994), 155–234.

[Dob68] Dobruschin, P., The description of a random field by means of conditional probabilities and conditions of its regularity, *Theory Probab. Appl.* 13 (1968), 197–224.

[DY21] Dudko, A. and Yampolsky, M., On computational complexity of Cremer Julia sets, *Fund. Math.* 252 (2021), 343–353.

[Fol13] Folland, G., *Real Analysis: Modern Techniques and Their Applications*, Wiley, 2013.

[GHR11] Galatolo, S., Hoyrup, M., and Rojas, C., Dynamics and abstract computability: compuing invariant measures, *Dyn. Syst.* 29 (2011).

[GHRS20] Gangloff, S., Herrera, A., Rojas, C., and Sablik, M., Computability of topological entropy: From general systems to transformations on Cantor sets and the interval, *Discrete Contin. Dyn. Syst.* 40 (2020).

[HP09] Haïssinsky, P. and Pilgrim, K. M., Coarse expanding conformal dynamics, *Astérisque* (2009).

[HT03] Hawkins, J. and Taylor, M., Maximal entropy measure for rational maps and a random iteration algorithm for Julia sets, *Int. J. Bifurc. Chaos* 13 (2003), 1442–1447.

[Hei01] Heinonen, J., *Lectures on Analysis on Metric Spaces*, Springer, New York, 2001.

[HM10] Hochman, M. and Meyerovitch, T., A characterization of the entropies of multidimensional shifts of finite type, *Ann. of Math. (2)* 171 (2010), 2011–2038.

[HR09] Hoyrup, M. and Rojas, C., Computability of probability measures and Martin-Löf randomness over metric spaces, *Inform. and Comput.* 207 (2009), 830–847.

[HS94] Hubbard, J. H. and Schleicher, D., "The spider algorithm", *Complex dynamical systems (Cincinnati, OH, 1994)*, vol. 49, Amer. Math. Soc., Providence, RI, 1994, pp. 155–180.

[KH95] Katok, A. and Hasselblatt, B., *Introduction to the Modern Theory of Dynamical Systems*, vol. 54, Cambridge Univ. Press, Cambridge, 1995.

[Li15] Li, Z., Weak expansion properties and large deviation principles for expanding Thurston maps, *Adv. Math.* 285 (2015), 515–567.

[Li17] Li, Z., *Ergodic Theory of Expanding Thurston Maps*, Atlantis Series in Dynamical Systems, Springer, 2017.

[Li18] Li, Z., Equilibrium states for expanding Thurston maps, *Comm. Math. Phys.* 357 (2018), 811–872.

[LS24] Li, Z. and Shi, X., *Entropy density and large deviation principles without upper semi-continuity of entropy*, 2024, arXiv: `2406.01712 [math.DS]`.

[Lyu82] Lyubich, M. Yu., The maximum-entropy measure of a rational endomorphism of the Riemann sphere, *Funct. Anal. Appl.* 16 (1982), 309–311.

[Mun00] Munkres, J., *Topology*, Featured Titles for Topology, Prentice Hall, Incorporated, 2000.

[Oli03] Oliveira, K., Equilibrium states for non-uniformly expanding maps, *Ergodic Theory Dynam. Systems* 23 (2003), 1891–1905.

[Prz90] Przytycki, F., On the Perron–Frobenius–Ruelle operator for rational maps on the Riemann sphere and for Hölder continuous functions, *Bol. Soc. Brasil. Mat.* 20 (1990), 95–125.

[PU10] Przytycki, F. and Urbański, M., *Conformal Fractals: Ergodic Theory Methods*, vol. 371, Cambridge Univ. Press, Cambridge, 2010.

[RSY20] Rafi, K., Selinger, N., and Yampolsky, M., Centralizers in mapping class groups and decidability of Thurston equivalence, *Arnold Math. J.* 6 (2020), 271–290.

[RY21] Rojas, C. and Yampolsky, M., Real quadratic Julia sets can have arbitrarily high complexity, *Found. Comput. Math.* 21 (2021), 59–69.

[Rue89] Ruelle, D., The thermodynamic formalism for expanding maps, *Comm. Math. Phys.* 125 (1989), 239–262.

[SY15] Selinger, N. and Yampolsky, M., Constructive geometrization of Thurston maps and decidability of Thurston equivalence, *Arnold Math. J.* 1 (2015), 361–402.

[Sin72] Sinai, Y. G., Gibbs measures in ergodic theory, *Russian Math. Surveys* 27 (1972), 21.

[Spa07] Spandl, C., Computing the topological entropy of shifts, *Math. Log. Q.* 53 (2007), 493–510.

[Vil09] Villani, C., *Optimal transport: old and new*, vol. 338, Springer-Verlag, Berlin, 2009.

[Wal75] Walters, P., A variational principle for the pressure of continuous transformations, *Amer. J. Math.* 97 (1975), 937–971.

[Wal82] Walters, P., *An Introduction to Ergodic Theory*, vol. 79, Springer, New York-Berlin, 1982.

[Wal92] Walters, P., Differentiability properties of the pressure of a continuous transformation on a compact metric space, *J. Lond. Math. Soc. (2)* 46 (1992), 471–481.

[Wei00] Weihrauch, K., *Computable analysis: an introduction*, Springer-Verlag, Berlin, Heidelberg, 2000.

[Yam21] Yampolsky, M., Towards understanding the theoretical challenges of numerical modeling of dynamical systems, *New Zealand J. Math.* 52 (2021), 453–467.

[Yur03] Yuri, M., Thermodynamic formalism for countable to one Markov systems, *Trans. Amer. Math. Soc.* 355 (2003), 2949–2971.

[Zie06] Ziegler, M., Effectively open real functions, *J. Complexity* 22 (2006), 827–849.

[Zin96] Zinsmeister, M., *Formalisme thermodynamique et systèmes dynamiques holomorphes*, vol. 4, Société Mathématique de France, 1996.

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